NUMBER THEORY.

204. Proposed by E. T. BELL, New York City.

Show that a necessary and sufficient condition that $6n + 1$ be a prime number is that no one of the quantities $(3m - r)/(2r + 1)$ for $r = 1, 2, 3, \cdots n - 1$ be an integer. Similarly for $6n - 1$, the quantities being $(3n - r)/(2r - 1)$ for $r = 2, 3, 4, \cdots n$.

SOLUTION BY ELIJAH SWIFT, Princeton, N. J.

Suppose first that the condition be not satisfied. Then we have for some $r < n$, $(3n - r)/(2r + 1) =$ an integer, say α, which is equivalent to the equation $6n + 1 = (2r + 1)2\alpha + 2r + 1$. Hence, $6n + 1$ has the factor $2r + 1$ and is not prime, so that the condition is necessary.

Again suppose that $6n + 1$ is not prime. We shall see that the given condition is not satisfied. In this case the odd number $6n + 1$ has two odd factors, which we will call $2r + 1$ and $2\alpha + 1$. Each must be as large as 5 since 3 is evidently not a factor of $6n + 1$. If $2\alpha + 1 \geqq 5$, $2r + 1 \leqq (6n + 1)/5 = n + (n + 1)/5$. But this is less than $2n - 1$ if $n \geqq 2$, and if $n = 1$, $6n + 1$ is prime. Hence $r \leqq n - 1$. Reversing the algebraic work above we infer from $6n + 1 = (2r + 1)(2\alpha + 1)$ the equation $(3n - r)/(2r + 1) = \alpha$. Hence the condition is sufficient.

The proof of the second half of the theorem is similar mutatis mutandis.

206. Proposed by R. D. CARMICHAEL, Indiana University.

Prove that the sum of the sixth powers of two integers cannot be the square of an integer.

SOLUTION BY ELIJAH SWIFT, Princeton, N. J.

We are required to show that the equation $a^6 + b^6 = c^2$, where a, b, c are integers, is impossible. We may suppose that a, b, and c are prime to each other, as any common factor may be divided out. Now all integral solutions of the equation $x^2 + y^2 = z^2$, where x, y, z are prime to each other are given by the formulas $x = 2mn$, $y = m^2 - n^2$, $z = m^2 + n^2$, where m and n are integers prime to each other, and one is even and one odd. We must, then, have $a^3 = 2mn$, $b^3 = m^2 - n^2$. Suppose that m is even and n odd. Since $2mn$ is a cube, and $2m$ and n are prime to each other, each must be a cube, and $2m$ must contain the factor 8, and m the factor 4. Call $2m = 8\alpha^3$, $n = \beta^3$. Substituting these values in the formula for b^3, we have $b^3 = 16\alpha^6 - \beta^6 = (4\alpha^3 - \beta^3)(4\alpha^3 + \beta^3)$. Since $2m$ and n have no common factor, $4\alpha^3$ and β^3 have none, and $4\alpha^3 - \beta^3$ and $4\alpha^3 + \beta^3$ must also be prime to each other. Hence each is a perfect cube, since their product is, and we have $4\alpha^3 + \beta^3 = q^3$, $4\alpha^3 - \beta^3 = p^3$.

But from these two equations follows at once by addition $8\alpha^3 = p^3 + q^3$ or $r^3 = p^3 + q^3$, which is impossible. If n is even and m is odd the same method applies.

This method is general and may be applied to prove that if a, b, c, and n are positive integers (A) $a^{2n} + b^{2n} = c^2$ is possible only if there exist positive integers p, q, and r such that (B) $p^n + q^n = r^n$. If we assume the truth of Fermat's